PROBLEMS FOR THE 15^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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CONTENTS

Keywords: 1. sequences, graphs – 2. combinatorial geometry, polygons – 3. algebra, polynomials – 4. game theory, polyominoes – 5. optimisation – 6. Diophantine equations, congruence – 7. matrices, primes – 8. percentage, coverings – 9. curves, locus, triangles – 10. extremal graph theory, bipartite graphs

Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}^n$	n-dimensional real space
$]a, b[$ and $[a, b]$	open and closed interval from a to b
$\det(A)$ or $\|A\|$	determinant of a matrix A
$\|x\|$	absolute value of x
$\#M$ or $\|M\|$	cardinality of a set M
$M \times N$	Cartesian product of two sets or graphs M and N
$M \cup N, M \cap N, M \setminus N$	union, intersection and difference of two sets M and N

1. Graphs of Sequences

Let $s = s_1, s_2, \ldots, s_k, \ldots$ be a sequence of positive integers, and let $n \in \mathbb{N}$ be a positive integer. Consider the graph $G = G_n(s)$ with set of vertices $V = \{1, 2, ..., n\}$ and the following set of edges: two vertices x and y of G are connected by an edge if and only if the number $P(x, y) = x + y$ is a member of the sequence s. The function P will be called the *parameter* of the graph G.

We would like to investigate this graph when s is:

a) an arithmetic sequence: $a_{k+1} = a_k + d$ for some $d \in \mathbb{N}$ and all $k \in \mathbb{N}$;
b) a geometric sequence: $a_{k+1} = a_k \cdot q$ for some $q \in \mathbb{N}$ and all $k \in \mathbb{N}$;
c) the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$;

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d) a recursive sequence: $a_{k+2} = \alpha a_{k+1} + \beta a_k$ for some $\alpha, \beta \in \mathbb{Z}$ and all $k \in \mathbb{N}$;
e) the sequence of primes $2, 3, 5, 7, 11, \dots$;
f) the sequence of binomial coefficients $\binom{k}{2}$;
g) another sequence of your choice;
h) an arbitrary sequence.

1. What is the number of connected components of G?

2. How many edges does G have?

3. What are possible lengths of cycles?

4. Study other properties of G.

5. Let now $P(x, y) = x \cdot y$. Answer the questions above for the corresponding graph G.

6. Same questions when $P(x, y) = ax^2 + bxy + cy^2$ is a quadratic polynomial.

7. Consider other functions as parameters.

8. Suggest and investigate other directions of research.

2. Counting Polygons

Let L be one of the following sets of points in the real plane:

a) the vertices of a regular polygon;
b) a circle;
c) a parabola;
d) $\mathbb{Z}^2$;
e) a triangular grid;
f) the entire $\mathbb{R}^2$.

For the following functions, give lower and/or upper bounds, and an asymptotic behaviour. Try to find an explicit formula or recurrence relations.

1. The maximum possible number of equilateral triangles among the polygons with vertices in arbitrary n points of the set L, denoted by $E_L(n)$.

2. The maximum possible number of isosceles triangles among the polygons with vertices in arbitrary n points of the set L, denoted by $I_L(n)$.

3. The maximum possible number of right triangles among the polygons with vertices in arbitrary n points of the set L, denoted by $R_L(n)$.

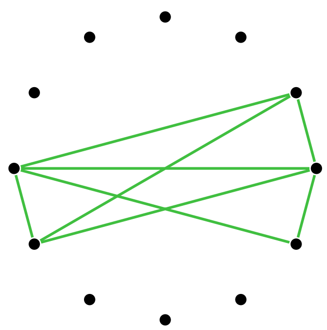

FIGURE 1. When L is the set of vertices of a regular 12-gon, one has $R_L(5) \geq 5$.

4. The maximum possible number of squares among the polygons with vertices in arbitrary n points of the set L, denoted by $S_L(n)$.

5. The maximum possible number of parallelograms among the polygons with vertices in arbitrary n points of the set L, denoted by $P_L(n)$.

6. Consider other similar functions.

7. Formulate and investigate the problem in higher dimensions.

3. **Divisibility of Polynomials**

1. Find all polynomials $P(x)$ with rational coefficients such that $P(x)$ divides the polynomial $Q(x) = P(x^2)$.

2. Find all polynomials $P(x)$ with rational coefficients such that $P(k)$ divides $P(k^2)$ for infinitely many positive integers k.

3. Let $m > 2$ be a positive integer. Characterise all polynomials $P(x)$ with rational coefficients such that $P(x)$ divides the polynomial $Q(x) = P(x^m)$.

4. Let s and t be distinct positive integers. Characterise all polynomials $P(x, y)$ with real coefficients such that $P(x^s, y)$ divides $P(x^t, y)$.

5. Let k and m be two distinct positive integers. Characterise all polynomials $P(x, y)$ that divide $P(x^k, y^m)$.

6. Investigate analogous questions for polynomials $P(x, y)$ with rational coefficients.

7. Investigate analogous questions for polynomials with coefficients in $\mathbb{Z}_n$.

8. Investigate analogous questions for polynomials with more than two variables.

9. Suggest and study other directions of research.

4. **A Game on Rectangles**

Let a, b and n be positive integers. Alicia and Benjamin are playing a game on an $a \times b$ rectangle. They alternately choose a polyomino of size n and place it on the rectangle without overlapping. (A *polyomino of size* n is a figure formed by joining n equal squares edge to edge.) The loser is the player who cannot place a polyomino anymore. Alicia begins.

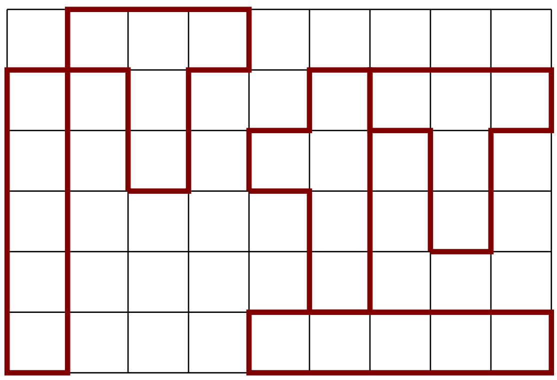

FIGURE 2. Placement of five polyominoes of size 5 on a 6×9 rectangle.

1. On which $a \times b$ rectangles does Alicia have a winning strategy? Consider the following cases:

a) $n = 2$ (there is only only type of polyominoes of size 2);
b) $n = 3$ (there are two types of polyominoes of size 3);
c) $n > 3$.

2. Now, let us modify the game and allow Alicia and Benjamin to choose any polyomino of size $1, 2, \ldots, n$. On which $a \times b$ rectangles does Alicia have a winning strategy?

3. Assume that it is up to Benjamin to choose the next polyomino that Alicia will have to place, and it is up to Alicia to choose the next polyomino that Benjamin will have to place. Who has a winning strategy? Consider the initial game and its variation.

4. Suggest and study additional directions of research.

5. **An Articulated Robot**

Martin, an articulated robot, spins a horizontal mechanical wheel which is divided into N equal sectors numbered from 0 to $N-1$ counterclockwise. An arrow is fixed on the ground. At the beginning it is pointing to the sector number 0 as shown in Figure 3.

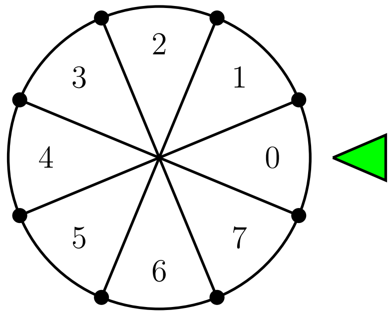

FIGURE 3. A mechanical wheel for $N = 8$ and an arrow.

Since Martin's joints are limited, he can only spin the wheel by a certain number of sectors to the right or to the left – by a_1 sectors, by a_2 sectors, ..., or by a_ℓ sectors, where $0 < a_1 < \ldots < a_\ell < N$ and $\ell < N$. Since Martin's batteries are limited, he can spin the wheel at most k times.

1. Suppose that $\ell = 1$ and $k \geq N$.

a) Given N and a_1, what is the largest number M on a sector that Martin can reach?
b) How many moves does it take, at the least, for Martin to reach the number M?
c) How many different numbers can Martin reach?

2. Solve the previous questions for any $\ell < N$.

3. Something is preventing the wheel from turning clockwise. Answer the previous questions when Martin can only rotate the wheel counterclockwise.

4. The wheel has been unlocked but Martin's batteries are almost empty. Martin can only spin the wheel at most $k < N$ times. Fortunately, he can now choose the numbers $a_1, \ldots, a_\ell \in \mathbb{N}$ as he wishes to better prepare his actions.

a) Given k and ℓ, estimate the maximum number of cells that Martin can reach over all possible choices of $a_1, \ldots, a_\ell \in \mathbb{N}$.
b) Estimate, as a function of k, the smallest value of ℓ for which Martin can choose $a_1, \ldots, a_\ell$ allowing him to reach all the sectors of the wheel.

5. Now, Martin can no longer choose in which direction to turn the wheel, and he is forced to make exactly $k = 2$ moves. Estimate the smallest value of ℓ for which Martin can choose the numbers $a_1, \ldots, a_\ell$ allowing him to reach all the sectors of the wheel if:

a) Martin must spin the wheel twice in the same direction;
b) Martin must spin the wheel once in one direction and once in the other.

6. Generalise and solve the previous question for $k > 2$.

7. Suggest and study other direction of research.

6. **Diophantine Equations**

1. Consider the congruence $y^2 \equiv x^3 + m \pmod{n}$ where m and $n \geq 2$ are given integers.

a) When $n \equiv 2 \pmod{3}$ is a prime number, show that the number of solutions $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$ does not depend on m.
b) In terms of m, classify all $n \in \mathbb{N}$ for which there is at least one solution $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

2. Consider the equation $y^2 + kb^2 = x^3 + a^3$. Determine all (or at least several) integers a and b such that the equation has no solution in non-zero integers x and y when

a) $k = 0$;
b) $k = 1$;
c) $k = 2$;
d) $k = -2$;
e) $k > 2$.

3. Find all integer solutions (x, y) of the equations:

a) $x^3 + 3x^2y - 3xy^2 - y^3 = 1$.
b) $y^2 = x^3 - 7$.

4. Suggest and study additional directions of research.

7. **Pseudoprimes of Matrices**

Let k and n be positive integers. Let A be an $n \times n$ matrix with integer entries. Consider the polynomial

$$f_{A,k}(x) = \det(A^k - xI) - \det(A - xI),$$

where I is the identity matrix of size n, and $\det(\cdot)$ denotes the determinant.

1. Prove that if k is prime, then all coefficients of $f_{A,k}(x)$ are divisible by k.

2. We say that a number k is a *pseudoprime for A* if k is composite and all coefficients of $f_{A,k}(x)$ are divisible by k.

a) Let $n \leq 3$. Among the $n \times n$ matrices A with non-negative entries and sum of entries at most 5, find the one(s) whose smallest pseudoprime is largest.
b) What matrices have infinitely many pseudoprimes?

3. Take $n = 2$. Consider the factorisation $k = \prod_{i=1}^{s} p_i^{\alpha_i}$ into primes. Denote

$$g(k) = \frac{1}{2}\prod_{i=1}^{s} p_i^{\alpha_i - 1}(p_i + 1) \quad \text{and} \quad h(k) = \frac{1}{2}\prod_{i=1}^{s} p_i^{\alpha_i - 1}(p_i - 1).$$

a) Is it true that for any A and k all coefficients of

$$\det(A^k - xI) - \det(A^{k-2h(k)} - xI)$$

are divisible by k?

b) Is it true that for any A and k all coefficients of

$$\det(A^{g(k)+h(k)} - xI) - \det(A^{g(k)-h(k)} - xI)$$

are divisible by k?

Give a proof or a counterexample.

4. Investigate the previous question for $n = 3$.

5. The well-known *test for prime numbers* based on Fermat's little theorem corresponds to the case $n = 1$ in the question 1. Try to find another test for prime numbers and generalise it appropriately to matrices. Then, formulate and study an analogue to the question 2..

6. Suggest and study additional directions of research.

8. **Mixing Cocktails**

At a birthday party, various milk cocktails mixed with n juices are being served. Every cocktail corresponds to a tuple of $n+1$ nonnegative real numbers $(a_0, a_1, \dots, a_n)$, where a_0 is percentage of the milk, and a_i is percentage of the i-th juice $(1 \leq i \leq n)$. Of course, $a_0 + a_1 + \cdots + a_n = 100$.

Let $f : \mathbb{R}^{n+1} \to [0, +\infty[$ be a nonnegative function. We will say that two cocktails $(a_0, a_1, \dots, a_n)$ and $(b_0, b_1, \dots, b_n)$ are ε-*close* if

$$f(a_0 - b_0, a_1 - b_1, \dots, a_n - b_n) < \varepsilon.$$

1. Suppose that $n = 2$.

a) How many different cocktails could there be at the party, if we know that no two cocktails (a_0, a_1, a_2) and (b_0, b_1, b_2) satisfy the inequality:

$$\max\{|a_0 - b_0|, |a_1 - b_1|, |a_2 - b_2|\} < 20?$$

b) How many different cocktails could there be at the party, if we know that no two cocktails (a_0, a_1, a_2) and (b_0, b_1, b_2) satisfy the inequality:

$$|a_0 - b_0| + |a_1 - b_1| + |a_2 - b_2| < 20?$$

2. Let ε be a positive real number. Denote by $d_n(\varepsilon)$ the maximum possible number of cocktails at a party, such that no two cocktails are ε-close. Find $d_n(\varepsilon)$ or give lower and upper bounds in the case that:

a) $f(x_0, x_1, \dots, x_n) = \max\{|x_0|, |x_1|, \dots, |x_n|\}$;
b) $f(x_0, x_1, \dots, x_n) = |x_0| + |x_1| + \cdots + |x_n|$.

You may start with $n = 2$ and 3.

3. Consider other interesting functions f.

4. Suggest and investigate additional directions of research.

9. **Triangles on Curves**

Recall that the second order curve is the set of points $\mathcal{R}$ in the real plane determined by a quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad \text{where} \quad a^2 + b^2 + c^2 \neq 0.$$

Let $\mathcal{R}$ be a second order curve such that there is a non-degenerate triangle whose vertices belong to it. Let A and B be two fixed points on $\mathcal{R}$.

Try to solve the questions 1-5 below. You can first consider the following simplifications of the problem:

a) Chose a particular curve and particular points on it (for example $x^2 + y^2 = 2$ and $A = (\sqrt{2}, 0), B = (1, 1)$).
b) Chose a particular curve and give an answer depending on the position of points A and B on it.

c) Solve the problem for a specific type of second order curves.

1. Describe the locus of incentres of non-degenerate triangles ABC with $C \in \mathcal{R}$.

2. Describe the locus of orthocentres of non-degenerate triangles ABC with $C \in \mathcal{R}$.

3. Describe the locus of circumcentres of non-degenerate triangles ABC with $C \in \mathcal{R}$.

4. Describe the locus of points of intersection of medians of non-degenerate triangles ABC with $C \in \mathcal{R}$.

5. Describe the locus of points of intersection of symmedians of non-degenerate triangles ABC with $C \in \mathcal{R}$.

6. Suggest and study additional directions of research.

10. **Maximal Number of Subgraphs**

Let H be a graph. Denote by $s_H(G)$ the number of subgraphs in a graph G which are isomorphic to H. Let $t_H(n)$ be the maximum of $s_H(G)$ over all graphs G with n vertices that are *triangle-free*:

$$t_H(n) = \max \{ s_H(G) \mid G \text{ is triangle-free and has } n \text{ vertices} \} .$$

We would like to understand for which graphs G this maximum is achieved, that is, $s_H(G) = t_H(n)$. Such graphs will be called *maximising for* H.

1. Let H be:

a) a path of length 1;
b) a four-cycle.

Find $t_H(n)$ and describe the maximising graphs G for H.

2. Are there any bipartite graphs H with a maximising graph which is not a complete bipartite graph?

3. Suppose that H is bipartite and has a perfect matching. Does it follow that any its maximising graph is a complete bipartite graph?

4. Can you come up with extra conditions on the graph H so that its maximising graph would necessary be a complete bipartite graph? For instance, additional assumption on H might be: having a matching on $n-2$ vertices, small radius, small diameter, large minimum degree, small maximum degree, etc.

5. What happens if we introduce extra assumptions on G? For instance, small diameter, large minimum degree, small maximum degree, contains no pentagons, etc. In other words, instead of $t_H(n)$, consider another function of maximisation of $s_H(G)$ over a subset of triangle-free graphs G with n vertices.

6. The same question for a combination of extra conditions on H and G.

7. Try to generalise the problem by taking K_r-free graphs instead of triangle-free graphs.

8. Suggest and study other directions of research.

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